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Did Beethoven Use the "Enneadecaphonic" Theory?

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ABSTRACT

The 19-tone equal temperament suggested since 1500 by great theoreticians, never accepted by classic composers (and now easily implemented, as well as systems of frequencies without musical value, by synthesizers), is shown to be the only system which would allow to formulate basic musical features (not explicitly appreciated centuries ago) in a non-ambiguous algorithmic theory useful to composers.

1. INTRODUCTION

Many classic-harmony rules are considered ambiguous, not only because in final analysis the human judgement of the composer is above every rule, but because unfortunately most basic musical values (e.g. *dissonant, atonal*) are confused in the traditional 12-tone temperament.

Details of an algorithm for composition of classic-like music in a different temperament have been presented in previous papers (Gagliardo, 1980; Gagliardo and Ghislandi, 1981). The present paper has a different point of view: first it describes *musical values* (i.e., basic notions of harmony) ranging from those discovered in the early history of music (such as the consonant intervals of *fifth* and *major third*) to those which later became important (like the freedom of modulations offered by *equal temperaments*) and towards *musical values* not explicitly pointed out by theoreticians but implicitly present in

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classic-music compositions (e.g., equivalence between four-consecutive-fifths and major-third modulation, non-equivalence between three-major-thirds modulation and unison, role of *atonal* chords). Then, among the multitude of possible temperaments considered in the history of music, we identify the one which would offer all these *musical values,* i.e., the temperament produced by partitioning the octave into 19 equal parts.

The idea of such a partition was considered centuries ago (by Zarlino in 1571, Praetorius in 1619, R. Smith in 1759, and others recently (Grove's dictionary; Mandelbaum and Chalmers, 1967) from the point of view of the approximation offered for basic intervals. It is remarkable that this partition has been considered when most of its possible *musical values* were not appreciated in either theory or practice.

Another reason for suggesting this partition in 19 tones to modern composers is that it makes possible a theory started in Gagliardo (1980) with the name of *enneadecaphonic.* So far we have only the definition of *atonal intervals* and a consequent classification of *admissible enneadecaphonic chords* and few optional rules for chord transitions (Gagliardo and Ghislandi, 1981), but the non-ambiguous correspondence between these notions and ear judgements seems to offer to an intelligent composer an actually useful beginning of a new theory of harmony. Computer programs can be easily based on this theory in many different ways: they essentially impose to reject notes which together with simultaneous notes would make a *non-admissible enneadecaphonic chord.*

The music composed with the help of this new theory, as long as people tolerate the traditional 12-tone temperament, can also be played on standard instruments no matter how tuned.

2. THE NATURAL SYSTEM

The first discovered musical values are the most consonant intervals:
interval of fifth = frequency quotient 3/2
interval of major third = frequency quotient 5/4
They are represented respectively by horizontal and vertical steps in the following scheme of the *natural system* of tones:

*	#A	*	*	*	*	*	*
*	#F	#C	#G	#D	α#A	*	*
*	βD	A	E	B	α#F	α#C	*
*	βbB	F	C	G	D	αA	*
*	βbG	βbD	bA	bE	bB	αF	*
*	*	*	*	*	bG	bD	*

(where # = 25/24, b = 24/25, α = "comma" = 81/80, β = 80/81).

In this scheme (which should be extended in the whole plane) each point represents all keys of the keyboard with the same name (i.e., frequencies x and $2^n x$ are identified; hence chord inversions are so far disregarded). Every triangle with the shape of

E   C   G
         or
C   G bE

is respectively a major or a minor chord. Every trapeze with the shape of

A   E   B   F   C   G   D
                         or
F   C   G   D bA bE bB

is respectively a major or a minor tonality.

In the *natural system* of tones one could have formulated a basic notion introduced in Gagliardo (1980): we call *tonal* the interval between two tones if there exists a tonality (i.e., trapeze) to which both tones belong; otherwise we call the interval *atonal.* For instance, a diminished fifth (like B-F) is *tonal,* while an augmented second (like bA-B) is *atonal.* As we will show, this notion, which seems finally to satisfy the modern search for *atonality*, plays a fundamental role in a new mathematical theory of harmony.

When the natural system was discovered, the feeling towards *atonality* was not yet developed, and therefore this possibility offered by the natural system was not realized.

Fig. 1. Growing rate of "atonal intervals" (marked by arrows) in the first 20 measures of "Moonlight Sonata" (Rhythm omitted).

3. EQUAL TEMPERAMENTS

Among the reasons which brought the natural system to a decline, many musicians remarked that the mathematical perfection of fifths and of major thirds is not what delights the musical ear, which might prefer some slightly less consonant intonation; furthermore the freedom of *modulation* (shift of tonality) would, when fidelity to the natural system were maintained, require the use of infinitely numerous tones within each octave, quite prohibitive for keyboard instruments.

The only way to keep finite the number of tones without limiting the freedom of modulation is obviously to indroduce a *tempered* partion of the octave with frequencies in geometric progression like the powers of $2^{1/n}$.

The choice of n (i.e., the number of equally spaced tones in each octave) is very critical. The historical choice $n = 12$, i.e., the *well tempered* system accepted by J. S. Bach and also completely accepted in the *dodecaphonic* music, gives a good approximation for the most primitive musical value: the interval of fifth (appreciated by Phytagoras); indeed, the seventh power of $2^{1/12}$

Fig. 2. E. Gagliardo: "La Camera dei Giocattoli". (Mostly atonal chords. Enneadecaphonic tuning required.)

is about 3/2. However, in the history of music many other *musical values* have been implicitly requested and gradually discovered although they were ignored and confused by the *well tempered* historical choice $n=12$.

4. POSITIVE AND NEGATIVE IDENTIFICATIONS

Every *equal temperament* (i.e., partition of the octave with frequencies proportional to the powers of $2^{1/n}$ for every given n) introduces several identifications between tones which would be slightly different in the natural system. We suggest considering such a feature from the point of view of a sequence of *modulations* which turns out to be identified with an unexpected (pleasant or unpleasant) coming back to the original tonality. It may be pleasant only if the actual error is very small, especially when partitioned in several steps. For instance, in equal temperaments with n not too large, in particular with $n = 12$, $n = 19$ and $n = 31$, four consecutive modulations down by a fifth followed by a modulation up by a major third bring to the original tonality with an error of a *comma* = 81/80 which especially in the $n=19$ and $n=31$ systems is well partioned being <1/3 of a comma each step (while for $n=12$ one has a double error in the final step); an outstanding example is given by the first 11 measures of the seventh symphony of Beethoven. Incidentally this is another strong reason for abandoning the natural system.

Among the three mentioned equal temperaments the traditional $n=12$ is the only one which allows a rather rude identification: three consecutive modulations by a major third bring to a tonality which is confused with the original: actual error 128/125 in only three steps.

To justify the historical choice $n=12$, we may remark that the *musical value* of pleasant identifications after remote modulations has never been explicitly pointed out in the history of music. Of course, the worse choice would not have been made by people aware of this notion.

5. ATONAL AND TONAL INTERVALS

As already remarked in section 2, the notions of atonal interval and of tonal interval introduced in Gagliardo (1980) could have been discovered within the natural system. They become meaningless in the traditional $n=12$ system since every couple of keys in a dodecaphonic keyboard belongs to some tonality. They keep sense, however, and they play a fundamental role in the $n=19$ as well as in the $n=31$ system.

The tones of the $n=19$ system, with frequencies proportional to the powers of $2^{1/19}$, can be called:

C, #C, *b*D, D, #D, *b*E, E, #E=*b*F, F, #F, *b*G, *b*G, G, #G, #G, *b*A, A, #A, *b*B, B, #B=*b*C

Representing fifths and major thirds respectively by horizontal and vertical steps we have the periodic scheme:

*	*	*	*	*	*	*	*
*	#A	#E=bF	#B=bC	bG	bD	*	
*	#F	#C	#G	#D	#A	*	
*	D	A	E	B	#F	*	
*	bB	F	C	G	D	*	
*	bG	bD	bA	bE	bB	*	
*	#D	#A	#E=bF	#B=bC	bG	*	
*	*	*	*	*	*	*	

A *restricted* version makes use only of the following 12 selected from the 19 tones:

C, bD, D, bE, E, F, #F, G, bA, A, bB, B

In this *restricted* version of the $n=19$ system the *atonal* intervals, according to the definition given in section 2, are:

A-b D, E-bA, B-bE, #F-bB, bD-#F *(diminished fourths)*
b D-E, b A-B, b E-#F, B-b D, #F-b A *(augmented seconds)*
b D-D, b E-E, F-#F, bA-A, bB-B *(chromatic semitones)*
as well as their inversions.

From a practical point of view the *restricted* version of the $n=19$ system amounts to just a different way of tuning a standard 12-key-keyboard instrument.

Analogous representations could be made for the $n=31$ system. Besides several minor reasons, the main point which suggests the $n=19$ system better than the $n=31$ is the difference between *atonal* and *tonal* intervals which turns out to be larger (hence easier to be ear detected) in the $n=19$ system; e.g., the *chromatic semitone* (like bE-E, F-#F) in the n 19 system turns out to be the half of the *diatonic semitone* (like E-F, B-C).

6. THE ENNEADECAPHONIC CHORDS

In the $n=19$ system as well as in its *restricted* version described in section 5, in terms of the notion of *atonal* and *tonal* interval we have the following definition:

We call *enneadecaphonic chord* a set of tones of the $n=19$ complete or restricted system, such that:

(1) There is at most one couple of tones joined by *atonal* interval.

(2) No two tones are joined by chromatic semitone (even though the two tones belong to different octaves).

In case of no atonal couple the chord is called *tonal;* in case of one it is called *atonal.*

When choosing, for each note of the chord, the actual octave to which the note must belong, one should avoid to *concentrate* two or three notes within respectively a diatonic semitone or a major third.

Optional rules for enneadecaphonic chord transitions are described in Gagliardo and Ghislandi (1981). If we play with an *enneadecaphonic*-tuned instrument the following sequences:

(i) Sequence of enneadecaphonic tonal chords:
*b*ACF, GCE, G*b*DF, *b*ACF,
*b*AC*b*E, GCE, G*b*BF, *b*ACF,
ACF, GDF, *b*BDF, ACF, *b*ACF,
GCE, G*b*DF, *b*ACF

(ii) Sequence of enneadecaphonic atonal chords:
DE*b*A, B*b*EG, *b*AC#F, *b*ABD,
*b*A*b*BE, #FC*b*E, *b*B*b*DE,
*b*BC#F, *b*AC#F, *b*ABF, *b*A*b*BE,
#FA*b*E, #F*b*AD, GB*b*E,
G*b*DE, B*b*DF

(iii) Sequence of enneadecaphonic illegal chords:
B*b*D*b*A, B*b*D*b*E, *b*A*b*B#F,
*b*AE#F, *b*D#F*b*A, *b*E#F*b*A,
*b*E#F*b*D, E#F*b*A

we very easily, detect respectively a soft, tense, meaningless feeling. If we play instead the three above sequences on a standard-tuned instrument, this strong difference is almost completely lost.

Therefore on a standard-tuned instrument the composer has neither this colourful variation of feelings nor a useful theory about it.

7. AN OPEN QUESTION

The first 30 measures of Beethoven's Moonlight Sonata can be translated in enneadecaphonic version: this can be done just by interpreting the differences which in the dodecaphonic system are *purely notational* (e.g., #B-C) as actual differences between two consecutive enneadecaphonic tones. The result of this translation (takes 15 of the 19 enneadecaphonic tones, and) contains chords which all satisfy our definition of enneadecaphonic chord.

This may seem rather strange because our definition is based on the notion of *atonal-tonal* interval, and therefore makes no sense in the 12-tone traditional temperament in which Beethoven wrote the Sonata.

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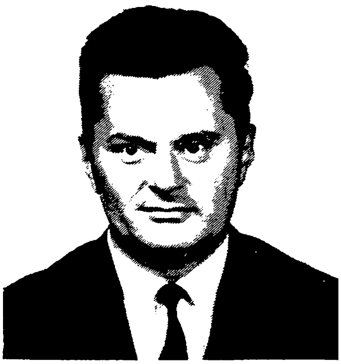

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